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Section A: Pure Mathematics

1 Show that, if n is an integer such that

$$(n-3)^3 + n^3 = (n+3)^3, \qquad (*)$$

then n is even and n^2 is a factor of 54. Deduce that there is no integer n which satisfies the equation $(*)$.

Show that, if n is an integer such that

$$(n-6)^3 + n^3 = (n+6)^3, \qquad (**)$$

then n is even. Deduce that there is no integer n which satisfies the equation $(**)$.

2 Use the first four terms of the binomial expansion of $(1-1/50)^{1/2}$, writing $1/50 = 2/100$ to simplify the calculation, to derive the approximation $\sqrt{2} \approx 1.414214$.

Calculate similarly an approximation to the cube root of 2 to six decimal places by considering $(1+N/125)^a$, where a and N are suitable numbers.

[You need not justify the accuracy of your approximations.]

3 Show that the sum S_N of the first N terms of the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots + \frac{2n-1}{n(n+1)(n+2)} + \cdots$$

is

$$\frac{1}{2}\left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2}\right).$$

What is the limit of S_N as $N \to \infty$?

The numbers a_n are such that

$$\frac{a_n}{a_{n-1}} = \frac{(n-1)(2n-1)}{(n+2)(2n-3)}.$$

Find an expression for a_n/a_1 and hence, or otherwise, evaluate $\sum_{n=1}^{\infty} a_n$ when $a_1 = \frac{2}{9}$.

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4 The integral I_n is defined by

$$I_n = \int_0^{\pi} (\pi/2 - x)\sin(nx + x/2)\,\mathrm{cosec}\,(x/2)\,\mathrm{d}x,$$

where n is a positive integer. Evaluate $I_n - I_{n-1}$, and hence evaluate I_n leaving your answer in the form of a sum.

5 Define the modulus of a complex number z and give the geometric interpretation of $|\,z_1 - z_2\,|$ for two complex numbers z_1 and z_2. On the basis of this interpretation establish the inequality

$$|\,z_1 + z_2\,| \leqslant |\,z_1\,| + |\,z_2\,|.$$

Use this result to prove, by induction, the corresponding inequality for $|\,z_1 + \cdots + z_n\,|$.

The complex numbers $a_1, a_2, \ldots, a_n$ satisfy $|a_i| \leqslant 3$ $(i = 1, 2, \ldots, n)$. Prove that the equation

$$a_1 z + a_2 z^2 \cdots + a_n z^n = 1$$

has no solution z with $|\,z\,| \leqslant 1/4$.

6 Two curves are given parametrically by

(1) $$x_1 = (\theta + \sin\theta), \quad y_1 = (1 + \cos\theta),$$

and

(2) $$x_2 = (\theta - \sin\theta), \quad y_2 = -(1 + \cos\theta).$$

Find the gradients of the tangents to the curves at the points where $\theta = \pi/2$ and $\theta = 3\pi/2$.

Sketch, using the same axes, the curves for $0 \leqslant \theta \leqslant 2\pi$.

Find the equation of the normal to the curve (1) at the point with parameter θ. Show that this normal is a tangent to the curve (2).

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7 Let

$$\begin{aligned} \mathrm{f}(x) &= \tan x - x, \\ \mathrm{g}(x) &= 2 - 2\cos x - x\sin x, \\ \mathrm{h}(x) &= 2x + x\cos 2x - \frac{3}{2}\sin 2x, \\ \mathrm{F}(x) &= \frac{x(\cos x)^{1/3}}{\sin x}. \end{aligned}$$

(i) By considering $\mathrm{f}(0)$ and $\mathrm{f}'(x)$, show that $\mathrm{f}(x) > 0$ for $0 < x < \pi/2$.

(ii) Show similarly that $\mathrm{g}(x) > 0$ for $0 < x < \pi/2$.

(iii) Show that $\mathrm{h}(x) > 0$ for $0 < x < \pi/4$, and hence that

$$x(\sin^2 x + 3\cos^2 x) - 3\sin x\cos x > 0$$

for $0 < x < \pi/4$.

(iv) By considering $\dfrac{\mathrm{F}'(x)}{\mathrm{F}(x)}$, show that $\mathrm{F}'(x) < 0$ for $0 < x < \pi/4$.

8 Points A, B, C in three dimensions have coordinate vectors **a, b, c**, respectively. Show that the lines joining the vertices of the triangle ABC to the mid-points of the opposite sides meet at a point R.

P is a point which is **not** in the plane ABC. Lines are drawn through the mid-points of BC, CA and AB parallel to PA, PB and PC respectively. Write down the vector equations of the lines and show by inspection that these lines meet at a common point Q.

Prove further that the line PQ meets the plane ABC at R.

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Section B: Mechanics

9 A light smoothly jointed planar framework in the form of a regular hexagon $ABCDEF$ is suspended smoothly from A and a weight 1kg is suspended from C. The framework is kept rigid by three light rods BD, BE and BF. What is the direction and magnitude of the supporting force which must be exerted on the framework at A?

Indicate on a labelled diagram which rods are in thrust (compression) and which are in tension.

Find the magnitude of the force in BE.

10 A wedge of mass M rests on a smooth horizontal surface. The face of the wedge is a smooth plane inclined at an angle α to the horizontal. A particle of mass m slides down the face of the wedge, starting from rest. At a later time t, the speed V of the wedge, the speed v of the particle and the angle β of the velocity of the particle below the horizontal are as shown in the diagram.

Let y be the vertical distance descended by the particle. Derive the following results, stating in (ii) and (iii) the mechanical principles you use:

(i) $V \sin\alpha = v \sin(\beta - \alpha)$;

(ii) $\tan\beta = (1 + m/M)\tan\alpha$;

(iii) $2gy = v^2(M + m\cos^2\beta)/M$.

Write down a differential equation for y and hence show that

$$y = \frac{gMt^2 \sin^2\beta}{2\,(M + m\cos^2\beta)}.$$

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11 A fielder, who is perfectly placed to catch a ball struck by the batsman in a game of cricket, watches the ball in flight. Assuming that the ball is struck at the fielder's eye level and is caught just in front of her eye, show that $\frac{\mathrm{d}}{\mathrm{d}t}(\tan\theta)$ is constant, where θ is the angle between the horizontal and the fielder's line of sight.

In order to catch the next ball, which is also struck towards her but at a different velocity, the fielder runs at constant speed v towards the batsman. Assuming that the ground is horizontal, show that the fielder should choose v so that $\frac{\mathrm{d}}{\mathrm{d}t}(\tan\theta)$ remains constant.

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Section C: Probability and Statistics

12 The diagnostic test AL has a probability 0.9 of giving a positive result when applied to a person suffering from the rare disease mathematitis. It also has a probability 1/11 of giving a false positive result when applied to a non-sufferer. It is known that only 1% of the population suffer from the disease. Given that the test AL is positive when applied to Frankie, who is chosen at random from the population, what is the probability that Frankie is a sufferer?

In an attempt to identify sufferers more accurately, a second diagnostic test STEP is given to those for whom the test AL gave a positive result. The probablility of STEP giving a positive result on a sufferer is 0.9, and the probability that it gives a false positive result on a non-sufferer is p. Half of those for whom AL was positive and on whom STEP then also gives a positive result are sufferers. Find p.

13 A random variable X has the probability density function

$$\mathrm{f}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geqslant 0, \\ 0 & x < 0. \end{cases}$$

Show that

$$\mathrm{P}(X > s + t \,|\, X > t) = \mathrm{P}(X > s).$$

The time it takes an assistant to serve a customer in a certain shop is a random variable with the above distribution and the times for different customers are independent. If, when I enter the shop, the only two assistants are serving one customer each, what is the probability that these customers are both still being served at time t after I arrive?

One of the assistants finishes serving his customer and immediately starts serving me. What is the probability that I am still being served when the other customer has finished being served?

14 The staff of Catastrophe College are paid a salary of A pounds per year. With a Teaching Assessment Exercise impending it is decided to try to lower the student failure rate by offering each lecturer an alternative salary of $B/(1 + X)$ pounds, where X is the number of his or her students who fail the end of year examination. Dr Doom has N students, each with independent probability p of failure. Show that she should accept the new salary scheme if

$$A(N + 1)p < B(1 - (1 - p)^{N+1}).$$

Under what circumstances could X, for Dr Doom, be modelled by a Poisson random variable? What would Dr Doom's expected salary be under this model?